STAR CONFIGURATIONS ON GENERIC HYPERSURFACES

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ABSTRACT. Let F be a homogeneous polynomial in $S = \mathbb{C}[x_0, \ldots, x_n]$. Our goal is to understand a particular polynomial decomposition of F; geometrically, we wish to determine when the hypersurface defined by F in $\mathbb{P}^n$ contains a star configuration. To solve this problem, we use techniques from commutative algebra and algebraic geometry to reduce our question to computing the rank of a matrix.

To A.V. Geramita on the occasion of his 70th birthday.

1. Introduction

Throughout this paper we work over the polynomial ring $S = \mathbb{C}[x_0, \ldots, x_n]$. Given any homogeneous polynomial $F \in S$ of degree d, one usually writes F as a sum of monomials of degree d, i.e., $F = \sum c_i m_i$, where $c_i \in \mathbb{C}$ and m_i is a monomial of degree d. However, different presentations are possible; for example, one can look for a sum of powers presentation of F, that is, find linear forms $\ell_1, \ldots, \ell_k$ such that

$$F = c_1\ell_1^d + c_2\ell_2^d + \cdots + c_k\ell_k^d.$$

Given a possible presentation, one can then ask many relevant questions about the presentation. Two such problems would be to find the minimal number of summands needed for the generic form, or for any given form, find an explicit presentation. Questions of this type were explored in the work [3, 7].

Presentations of F can also be reinterpreted as geometric questions. As an example, if F has a sum of powers presentation as above, then F is an element of the ideal $I = (\ell_1, \ldots, \ell_k)$. But this means that the hypersurface defined by F in $\mathbb{P}^n$ contains the variety defined by I. A presentation question could therefore be reformulated as asking if a (generic) hypersurface contains a special subvariety. This type of question has a long history, e.g., a number of authors have investigated the question of when a hypersurface contains a complete intersection (the papers [4, 5, 12, 14, 15, 17] form a partial list of papers devoted to this topic).

In this paper, we want to investigate the following type of polynomial decomposition. To state our question, we use the notation $[l] = \{1, \ldots, l\}$.

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Question 1.1. *For which tuples $(n, l, r, d) \in \mathbb{N}_+^4$ is it possible to present a generic homogeneous form F of degree d in $n+1$ variables as*

$$F = \sum_{\substack{\sigma = \{i_1, \ldots, i_r\} \subseteq [l], \\ |\sigma| = r}} L_\sigma M_\sigma \tag{1.1}$$

where $L_\sigma = L_{i_1} L_{i_2} \cdots L_{i_r}$, $\{L_1, \ldots, L_l\}$ are generic linear forms, and M_σ is a form of degree $d - r$.

In order to express F in the form (1.1), we immediately notice some simple restrictions on the tuples (n, l, r, d), namely $r \leq l$ and $r \leq d$. The goal of this paper is to give an almost complete answer to this question. Our main result is:

Theorem 1.2. *Let $(n, l, r, d) \in \mathbb{N}_+^4$ be such that $r \leq \min\{d, l\}$.*

1. *If $l - r + 1 < n$ and $d \gg 0$, then the generic degree d form in $n+1$ variables cannot be written in the form (1.1).*
2. *If $l - r + 1 = n$, then the generic degree d form in $n+1$ variables can be written in the form (1.1) if and only if (n, l, r, d) belongs to the following list:*
   - *(i) $(n, l, r, d) = (1, l, l, d)$ for all $d \geq l \geq 1$, or*
   - *(ii) $(n, l, r, d) = (2, 2, 1, d)$ for all $d \geq 1$, or*
   - *(iii) $(n, l, r, d) = (2, 3, 2, d)$ for all $d \geq 2$, or*
   - *(iv) $(n, l, r, d) = (2, 4, 3, d)$ for all $d \geq 3$, or*
   - *(v) $(n, l, r, d) = (2, 5, 4, d)$ for all $d \geq 5$, or*
   - *(vi) $(n, l, r, d) = (n, n, 1, d)$ for all $n \geq 3$ and $d \geq 1$, or*
   - *(vii) $(n, l, r, d) = (n, n+1, 2, d)$ for all $n \geq 3$ and $d \geq 2$, or*
   - *(viii) $(n, l, r, d) = (n, n+2, 3, d)$ for all $n \geq 3$ and $d \geq 3$.*
3. *If $l - r + 1 > n$, then every degree d form (not just the generic one) in $n+1$ variables can be written in the form (1.1).*

Geometrically, Question 1.1 is asking if the generic degree d hypersurface contains a star configuration (see the definition in the next section). Question 1.1 was studied in the case that $n = 2$ by the first and third author in [6]. We refer the reader to this paper for the statements of Theorem 1.2 involving $n = 2$. Note that our answer is almost complete since there may be tuples (n, l, r, d) with $l - r + 1 < n$ with d small enough such that Question 1.1 has a positive answer. However, we currently know of no such examples.

Our paper is structured as follows. In the next section, we give two interpretations of Question 1.1: an algebraic version and a geometric version. The geometric version of this question asks about star configurations on hypersurfaces. We also prove some cases of Theorem 1.2. In Section 3, we look at non-existence results, that is, look for ways to eliminate various (n, l, r, d) from consideration. By the end of Section 3, we will have proved all of Theorem 1.2 except the part of statement (2) involving the tuples $(n, n+2, 3, d)$. The remainder of the paper is devoted to the proof of this case. In Section 4, we translate our question again. The new translation reduces our question to showing that a specific evaluation matrix has maximal rank. We then answer this corresponding linear algebra question in Sections 5 and 6.

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2. Star Configurations

We reformulate Question 1.1 as an algebraic question and a geometric question. To state the geometric counterpart, we will introduce star configurations.

We begin with the algebraic reformulation. In $S = \mathbb{C}[x_0, \ldots, x_n]$, let $L_1, \ldots, L_l$ be l linear homogeneous forms. We let $[l] = \{1, \ldots, l\}$ and we set

$$L_\sigma := L_{i_1} L_{i_2} \cdots L_{i_r} \text{ for any } \sigma = \{i_1, \ldots, i_r\} \subseteq [l].$$

We shall write $V(L)$ to mean the hypersurface in $\mathbb{P}^n$ defined by L. If L is linear, then $V(L)$ is usually called a **hyperplane**. We say that the l linear homogeneous forms $L_1, \ldots, L_l$ are **general linear forms** if any $n+1$ of the linear forms are linearly independent. If $l < n+1$, then we require that the l linear forms are linearly independent.

The algebraic reformulation of Question 1.1 is an ideal membership problem.

Question 2.1 (Algebraic Question)**.** *Fix a tuple $(n, l, r, d) \in \mathbb{N}_+^4$ with $r \leq \min\{d, l\}$. Given a generic homogeneous form $F \in S = \mathbb{C}[x_0, \ldots, x_n]$ of degree d, is it possible to find l general linear forms $L_1, \ldots, L_l$ such that $F \in I = (L_\sigma \mid \sigma \subseteq [l]$ and $|\sigma| = r)$?*

The geometric interpretation of Question 1.1 is in terms of star configurations.

Definition 2.2. Let $L_1, \ldots, L_l$ be l general linear forms in $S = \mathbb{C}[x_0, \ldots, x_n]$. Let $r \leq l$ be any positive integer. The **star configuration** of type (l, r), denoted $\mathbb{X}(l, r)$, is the algebraic variety of $\mathbb{P}^n$ defined by the homogeneous ideal

$$J = \bigcap_{\substack{\tau = \{j_1, \ldots, j_{l-r+1}\} \subseteq [l] \\ |\tau| = l-r+1}} (L_{j_1}, \ldots, L_{j_{l-r+1}}).$$

Equivalently, the algebraic variety $\mathbb{X}(l, r) = V(J) \subset \mathbb{P}^n$ is the union of all the linear spaces obtained by intersecting $l-r+1$ of the hyperplanes $\{L_i = 0\}$ in all possible ways.

The name "star configuration" was first suggested by A.V. Geramita because a star configuration $\mathbb{X}(5, 4) \subseteq \mathbb{P}^2$ resembles a star drawn with five lines. Star configurations have proven to be interesting varieties, in part, because they exhibit some nice extremal behavior. To date, much of the research (see [1, 2, 6, 8, 9, 10, 13, 16]) has focused on the case of star configurations of the type $(l, l-n+1)$; in this case, $\mathbb{X}(l, l-n+1)$ is a finite set of points. This fact, and others, will follow from the next lemma which recalls some of the relevant properties of star configurations for this project.

Lemma 2.3. *Let $L_1, \ldots, L_l$ be l general linear forms of $S = \mathbb{C}[x_0, \ldots, x_n]$ and $0 < r \leq l$.*

- *(i) If $l - r + 1 > n$, then $\mathbb{X}(l, r) = \emptyset$.*
- *(ii) If $l - r + 1 \leq n$, then $\dim \mathbb{X}(l, r) = n - (l - r + 1)$.*
- *(iii) If $l - r + 1 = n$, then $\mathbb{X}(l, r)$ is a set of $\binom{l}{n}$ distinct points.*

(iv) If $l-r+1 \leq n$, *then* $I_{\mathbb{X}(l,r)} = (L_\sigma \mid \sigma \subseteq [l]$ *and* $|\sigma| = r)$.

Proof. (i) If $l-r+1 > n$, then for any $\tau \subseteq [l]$ with $|\tau| = l-r+1$, the ideal $(L_{j_1}, \ldots, L_{j_{l-r+1}})$ must be the irrelevant ideal because the L_i's are general linear forms. Consequently

$$\mathbb{X}(l,r) = V(J) = \bigcup_{\tau \subseteq [l],\ |\tau| = l-r+1} V((L_{j_1}, \ldots, L_{j_{l-r+1}})) = \emptyset.$$

(ii) This fact follows directly from the definition of $J = I_{\mathbb{X}(l,r)}$ and from the fact that the L_i's are general linear forms.

(iii) By (ii), $\mathbb{X}(l,r)$ is zero-dimensional. For any $\tau \subseteq [l]$ with $|\tau| = l-r+1 = n$, the ideal $(L_{j_1}, \ldots, L_{j_n})$ defines a point in $\mathbb{P}^n$. There are then $\binom{l}{n}$ such ideals, each defining a different point.

(iv) Let I denote the ideal on the right in the statement. We first show $I \subseteq I_{\mathbb{X}(l,r)}$. Take any generator of I, say L_σ for some $\sigma \subseteq [l]$. We claim that for any subset $\tau = \{j_1, ..., j_{l-r+1}\} \subseteq [l]$, the generator $L_\sigma \in (L_{j_1}, ..., L_{j_{l-r+1}})$. This claim follows once we note that $\sigma \cap \tau \neq \emptyset$. Indeed, if these two sets were disjoint, then $|\sigma \cup \tau| = r+(l-r+1) = l+1 > l$, which contradicts the fact that $[l]$ has only l distinct elements. So, each generator of I belongs to $I_{\mathbb{X}(l,r)}$, thus showing one inclusion.

For the reverse inclusion, we do induction on the tuple (l,r). If $r = 1$ and for any integer $1 = r \leq l$,

$$I_{\mathbb{X}(l,1)} = \bigcap_{\substack{\tau \subseteq [l] \\ |\tau| = l-1+1}} (L_{j_1}, \ldots, L_{j_{l-1+1}}) = (L_1, \ldots, L_l) = (L_\sigma \mid \sigma \subseteq [l] \text{ and } |\sigma| = 1).$$

In the case that $r = l$, we have

$$\begin{aligned} I_{\mathbb{X}(l,l)} &= \bigcap_{\substack{\tau \subseteq [l] \\ |\tau| = l-l+1 = 1}} (L_{j_1}, \ldots, L_{j_{l-l+1}}) = (L_1) \cap (L_2) \cap \cdots \cap (L_l) = (L_1 \cdots L_l) \\ &= (L_\sigma \mid \sigma \subseteq [l] \text{ and } |\sigma| = l). \end{aligned}$$

So, statement (iv) is true for all tuples of the form $(l,1)$ and (l,l).

So, for the induction step, let (l,r) be any tuple with $1 < r < l$. We then have

$$\begin{aligned} I_{\mathbb{X}(l,r)} &= \bigcap_{\substack{\tau = \{j_1, \ldots, j_{l-r+1}\} \subseteq [l] \\ |\tau| = l-r+1}} (L_{j_1}, \ldots, L_{j_{l-r+1}}) \\ &= \bigcap_{\substack{\tau = \{j_1, \ldots, j_{l-r+1}\} \subseteq [l] \\ |\tau| = l-r+1 \text{ and } l \in \tau}} (L_{j_1}, \ldots, L_{j_{l-r+1}}) \cap \bigcap_{\substack{\tau = \{j_1, \ldots, j_{l-r+1}\} \subseteq [l] \\ |\tau| = l-r+1 \text{ and } l \notin \tau}} (L_{j_1}, \ldots, L_{j_{l-r+1}}) \\ &= \bigcap_{\substack{\tau = \{j_1, \ldots, j_{l-r-1+1}\} \subseteq [l-1] \\ |\tau| = (l-1)-r+1}} (L_{j_1}, \ldots, L_{j_{l-r-1+1}}, L_l) \cap \bigcap_{\substack{\tau = \{j_1, \ldots, j_{l-r+1}\} \subseteq [l-1] \\ |\tau| = (l-1)-(r-1)+1}} (L_{j_1}, \ldots, L_{j_{l-r+1}}) \\ &= (I_{\mathbb{X}(l-1,r)}, L_l) \cap I_{\mathbb{X}(l-1,r-1)}. \end{aligned}$$

If we apply our induction hypothesis, we get

$$\begin{aligned}(I_{\mathbb{X}(l-1,r)}, L_l) \cap I_{\mathbb{X}(l-1,r-1)} &= ((L_\sigma \mid \sigma \subseteq [l-1] \text{ and } |\sigma| = r), L_l) \cap (L_\sigma \mid \sigma \subseteq [l-1] \text{ and } |\sigma| = r-1)\\ &\subseteq (L_\sigma \mid \sigma \subseteq [l-1] \text{ and } |\sigma| = r) + L_l(L_\sigma \mid \sigma \subseteq [l-1] \text{ and } |\sigma| = r-1)\\ &= (L_\sigma \mid \sigma \subseteq [l] \text{ and } |\sigma| = r) = I.\end{aligned}$$

Since we have already shown that $I \subseteq I_{\mathbb{X}(l,r)}$ for all (l, r), the desired result now follows. □

Remark 2.4. We can find an alternative proof of Lemma 2.3, (iv), in [9, Proposition 2.9].

When $r = l - n + 1$, Lemma 2.3 implies that $\mathbb{X}(l, l-n+1) \subseteq \mathbb{P}^n$ is a collection of $\binom{l}{n}$ points. In this case, we can compute the corresponding Hilbert function.

Theorem 2.5. *Let $\mathbb{X}(l, l-n+1) \subset \mathbb{P}^n$ be a star configuration. Then $\mathbb{X}(l, l-n+1)$ has the Hilbert function of $\binom{l}{n}$ generic points, that is,*

$$HF(\mathbb{X}(l, l-n+1), t) = \dim_{\mathbb{C}}(S/I_{\mathbb{X}(l,l-n+1)})_t = \min\left\{\binom{n+t}{n}, \binom{l}{n}\right\}.$$

Furthermore, the ideal $I_{\mathbb{X}(l,l-n+1)}$ is generated by $\binom{l}{n-1}$ forms of degree $l - n + 1$.

Proof. The Hilbert function of a finite set of points $\mathbb{X}$ is a non-decreasing sequence that stabilizes at $|\mathbb{X}|$, so $HF(\mathbb{X}(l, l-n+1), t) \leq \binom{l}{n}$ for all t. From Lemma 2.3 (iv), because $I_{\mathbb{X}(l,l-n+1)}$ is generated in degree $l - n + 1$, then $(I_{\mathbb{X}(l,l-n+1)})_t = (0)$ for all $t < l - n + 1$, whence $\dim_{\mathbb{C}}(S/I_{\mathbb{X}(l,l-n+1)})_t = \dim_{\mathbb{C}} S_t = \binom{t+n}{t}$. The conclusion now follows from the fact that $\binom{l}{n} = \binom{t+n}{n}$ when $t = l - n$. The second statement follows from [11, Proposition 4] since $\mathbb{X}(l, l-n+1)$ has the Hilbert function of $\binom{l}{n}$ generic points. □

We can use Theorem 2.5 to prove the following result.

Theorem 2.6. *Let $L_1, \ldots, L_l$ be l general linear forms of $S = \mathbb{C}[x_0, \ldots, x_n]$ and $0 < r \leq l$. If $l - r + 1 > n$, then*

$$(L_\sigma \mid \sigma \subseteq [l] \text{ and } |\sigma| = r) = (S_r).$$

Proof. Set $I = (L_\sigma \mid \sigma \subseteq [l] \text{ and } |\sigma| = r)$. Because $\dim_{\mathbb{C}} S_r = \binom{r+n}{n}$, the result will follow if we can find a set of $\binom{r+n}{n}$ linearly independent generators of I.

Let $L_1, \ldots, L_{n+r-1}$ be the first $n + r - 1$ forms of $L_1, \ldots, L_l$ (since $l > n + r - 1$, there is at least one more form L_{n+r}). If we set

$$I' = (L_\sigma \mid \sigma \subseteq [n+r-1] \text{ and } |\sigma| = r),$$

then I' is the defining ideal of a star configuration $\mathbb{X}(n + r - 1, r)$. In particular, since $(n + r - 1) - r + 1 = n$, $\mathbb{X}(n + r - 1, r)$ is a set of $\binom{n+r-1}{n}$ points.

By Theorem 2.5, the ideal $I_{\mathbb{X}(n+r-1,r)}$ is generated by $\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$ linearly independent elements of degree r. Let

$$A = \{L_\sigma \mid \sigma \subseteq [n+r-1] \text{ and } |\sigma| = r\}$$

be these generators. Now consider the set of generators of I of the form:

$$B = \{L_\tau L_{n+r} \mid \tau \subseteq [n+r-1] \text{ and } |\tau| = r-1\}.$$

It follows that $|B| = \binom{n+r-1}{r-1} = \binom{n+r-1}{n}$. Then $|A \cup B| = \binom{n+r-1}{n-1} + \binom{n+r-1}{n} = \binom{n+r}{n}$. So, we will be finished if we can show that the elements of $A \cup B$ are linearly independent.

Suppose, for a contradiction, that there was some linear combination

$$\sum_{L_\sigma \in A} c_\sigma L_\sigma + \sum_{L_\tau L_{n+r} \in B} d_\tau L_\tau L_{n+r} = 0$$

with $c_\sigma, d_\tau \in \mathbb{C}$, not all zero. There must be at least one nonzero d_τ since all the elements of A are linear independent. Rearranging the above equation gives:

$$\sum_{L_\tau L_{n+r} \in B} d_\tau L_\tau L_{n+r} \in I' = I_{\mathbb{X}(n+r-1,r)}.$$

Assume that $d_\tau \neq 0$. If $\tau = \{i_1, \ldots, i_{r-1}\}$, then $[n+r-1] \setminus \tau = \{j_1, \ldots, j_n\}$. Let P be the point of $V(I') = \mathbb{X}(n+r-1,r)$ defined by $(L_{j_1}, \ldots, L_{j_n})$. Because the L_is are general linear forms, the point P does not vanish at any of $L_{i_1}, \ldots, L_{i_{r-1}}, L_{n+r}$. On the other hand, for any $\tau \neq \tau' \subseteq [n+r-1]$ with $|\tau'| = r-1$, we must have $\tau' \cap \{j_1, \ldots, j_n\} \neq \emptyset$, and thus P vanishes at $L_{\tau'} L_{n+r}$. We then have

$$\left(\sum_{L_\tau L_{n+r} \in B} d_\tau L_\tau L_{n+r} \right)(P) = d_\tau L_\tau(P) L_{n+r}(P) = 0.$$

But since $L_\tau(P) \neq 0$ and $L_{n+r}(P) \neq 0$, we must have $d_\tau = 0$, a contradiction. □

The above theorem will be key in proving Theorem 1.2 (3), i.e., when $l - r + 1 > n$. When $l - r + 1 \leq n$, Question 1.1 can be geometrically reinterpreted:

Question 2.7 (Geometric Question)**.** *Let l, r, and d be positive integers such that $r \leq \min\{d, l\}$ and $l - r + 1 \leq n$. For a generic homogeneous form $F \in S = \mathbb{C}[x_0, \ldots, x_n]$ of degree d, is there a star configuration $\mathbb{X}(l, r)$ such that $\mathbb{X}(l, r) \subseteq V(F)$?*

We answer Question 2.7 for two trivial cases.

Lemma 2.8. *Let l, r, and d be positive integers such that $r \leq l$ and $r \leq d$. Furthermore, suppose that $l - r + 1 = n$.*

- (i) *If $l = n$ (and thus, $r = 1$) then every generic hypersurface of degree $d \geq 1$ contains a star configuration $\mathbb{X}(l, 1)$.*
- (ii) *If $l = n+1$ (and thus $r = 2$), then every generic hypersurface of degree $d \geq 2$ contains a star configuration $\mathbb{X}(l, 2)$.*

Proof. (i) Every hypersurface contains a point, which can be viewed as a $\mathbb{X}(l, 1)$.

(ii) In this case $\mathbb{X}(l, 2)$ is $\binom{n+1}{n} = n+1$ points in general linear position, and every generic hypersurface of degree $d \geq 2$ contains such a configuration of points. □

We now pause and prove part of Theorem 1.2.

Proof of Theorem 1.2 (2), cases (i) to (vii). As an opening remark, we can eliminate any tuple (n, l, r, d) that has $d < r$ or $d < l$. As mentioned in the introduction, the statements

are true for all tuples with $n = 2$, as proved in [6]; we refer the reader to this paper for these proofs.

We now consider the case that $n = 1$, and consequently, $l - r + 1 = 1$ implies that $l = r$. Consider all the tuples of the form $(1, l, l, d)$. Since $l = r$, and we must have $d \geq l$, we can omit all tuples with $d < l$. So, it suffices to show that Question 1.1 has a positive answer with $n = 1$ if and only if $(n, r, l, d) = (1, l, l, d)$ with $d \geq l \geq 1$. So let us first suppose there are general linear forms $L_1, \ldots, L_l$ such that $F \in I = (L_\sigma \mid \sigma \subseteq [l] \text{ and } |\sigma| = r = l) = (L_1 \cdots L_l)$. Because $\deg F = d \geq r = l$, we have that $(n, l, r, d) = (1, l, l, d)$ with $d \geq l = r \geq 1$. For the converse, suppose we are given a generic form F of degree d. Because $F \in \mathbb{C}[x_0, x_1]$, we can factor F as $F = L_1 L_2 \cdots L_d$. Because F is generic, we can assume that each L_i has multiplicity one. Since $d \geq l \geq 1$ and because $l = r$, we can take our general linear forms to be $L_1, \ldots, L_l$. In this case $F \in I = (L_\sigma \mid \sigma \subseteq [l] \text{ and } |\sigma| = r) = (L_1 L_2 \cdots L_r)$. Thus Question 1.1 has a positive answer.

Furthermore, Lemma 2.8 implies that Question 1.1 has a positive answer if $(n, l, r, d) = (n, n, 1, d)$ for all $n \geq 3$ and $d \geq 1$ and if $(n, l, r, d) = (n, n + 1, 2, d)$ for all $n \geq 3$ and $d \geq 2$. □

Proof of Theorem 1.2, (3). Suppose that $(n, l, r, d) \in \mathbb{N}_+^4$ with $r \leq \min\{d, l\}$. Suppose that $l - r + 1 > n$ and F is any homogeneous form of F of degree d. Let $L_1, \ldots, L_l$ be any general linear forms. Then by Theorem 2.6, $I = (L_\sigma \mid \sigma \subseteq [l] \text{ and } |\sigma| = r) = (S_r)$. So, $F \in S_d \subseteq I$, and thus, by Question 2.1, Question 1.1 has a positive answer. □

We continue with the proof of Theorem 1.2 at the end of the next section.

3. Non-existence answers

We now give negative answers to Question 1.1 in a number of cases, allowing us to reduce Question 1.1 to one non-trivial case, which will be studied in the remaining sections.

We first provide an asymptotic negative answer to Question 1.1 when $l - r + 1 < n$.

Lemma 3.1. *If $l - r + 1 < n$ and $d \gg 0$, then the generic degree d hypersurface does not contain a star configuration $\mathbb{X}(l, r)$.*

Proof. Let $\mathbb{P}S_d$ be the parameter space for degree d hypersurfaces in $\mathbb{P}^n$. Also, let $\mathcal{H} \subset (\check{\mathbb{P}^n})^l$ be the parameter space for star configurations $\mathbb{X}(l, r)$ in $\mathbb{P}^n$. Consider the incidence correspondence

$$\Sigma_{d,l,r} = \{(H, \mathbb{X}(l, r)) : \mathbb{X}(l, r) \subset H\} \subset \mathbb{P}S_d \times \mathcal{H}$$

and the natural projection maps

$$\psi_{d,l,r} : \Sigma_{d,l,r} \longrightarrow \mathcal{H} \text{ and } \phi_{d,l,r} : \Sigma_{d,l,r} \longrightarrow \mathbb{P}S_d.$$

We have that $\phi_{d,l,r}$ is dominant if and only if Question 1.1 has an affirmative answer.

Using a standard fibre dimension argument, if $d \geq l - n + 1$, then we get

$$\dim \Sigma_{d,l,r} \leq \dim \mathcal{H} + \dim_{\mathbb{C}}(I_{\mathbb{X}(l,r)})_d - 1 = \dim \mathcal{H} + \binom{n+d}{d} - \binom{l}{n} - 1.$$

Hence we have that

$$\dim \Sigma_{d,l,r} - \dim \mathbb{P}S_d \leq \dim \mathcal{H} + \dim_{\mathbb{C}}(I_{\mathbb{X}(l,r)})_d - \binom{d+1}{n} = \dim \mathcal{H} - HF(\mathbb{X}(l,r), d).$$

Now $\dim \mathcal{H} = ln$ and $HF(\mathbb{X}(l,r), d)$ is an eventually positive polynomial in d of degree $n-(l-r+1)$ by Lemma 2.3 (ii). Thus, for $d \gg 0$, the map $\phi_{d,l,r}$ cannot be dominant. $\square$

We now restrict to the case that $l - r + 1 = n$. In light of Question 2.7, we are asking if the generic hypersurface contains a star configuration $\mathbb{X}(l, r)$. Because $l - r + 1 = n$, l determines r, so we will simplify our notation slightly and write $\mathbb{X}(l) \subseteq \mathbb{P}^n$ for $\mathbb{X}(l, r)$. We can now eliminate "large" values of d when $l - r + 1 = n$.

Theorem 3.2. *If $n > 2$ and $l > n + 2$, then the generic degree d hypersurface in $\mathbb{P}^n$ does not contain a star configuration $\mathbb{X}(l)$ for any d. If $n = 2$, then the generic degree d plane curve does not contain a star configuration $\mathbb{X}(l)$ for $l > 5$ and any d.*

Proof. The case $n = 2$ is [6, Theorem 3.1], so we only consider the case $n > 2$. We use the notation of Lemma 3.1 dropping the unnecessary subindex r. Using a standard fibre dimension argument, if $d \geq l - n + 1$, then

$$\dim \Sigma_{d,l} \leq \dim \mathcal{H} + \dim_{\mathbb{C}}(I_{\mathbb{X}(l)})_d - 1 = \dim \mathcal{H} + \binom{n+d}{d} - \binom{l}{n} - 1.$$

Note that we use Theorem 2.5 to compute $\dim_{\mathbb{C}}(I_{\mathbb{X}(l)})_d$. Thus, the answer to our question is affirmative only if $\dim \Sigma_{d,l} \geq \dim \mathbb{P}S_d$, that is, only if

$$ln - \binom{l}{n} \geq 0. \tag{3.1}$$

We show that (3.1) does not hold if $l \geq n + 3$. If $l = n + 3$, then (3.1) yields

$$n(n+3) - \binom{n+3}{n} = -(n+3)\frac{n^2 - 3n + 2}{6} \geq 0,$$

and this does not hold for $n > 2$. So suppose that $l > n + 3$. We then have

$$\begin{aligned} ln - \binom{l}{n} &= ln - \frac{l(l-1)\cdots(l-n+1)}{n!} \\ &\geq \frac{n+3}{n!}[n(n!) + (1-l)(l-2)(l-3)\cdots(l-n+1)] \\ &\geq \frac{n+3}{n!}[n(n!) + (1-l)(n+1)n\cdots 4] \\ &= \frac{n+3}{n!}\left[n(n!) + (1-l)\frac{(n+1)!}{6}\right] \\ &= \frac{n+3}{n!}\left[n(n!) + (1-l)\frac{(n+1)(n!)}{6}\right] = (n+3)\left[n + (1-l)\frac{(n+1)}{6}\right] \geq 0. \end{aligned} \tag{3.2}$$

But (3.2) is true only if $\frac{7n+1}{n+1} \geq l > n + 3$ and this is a contradiction for $n > 2$. $\square$

We use the results of this section to continue our proof of Theorem 1.2.

Proof of Theorem 1.2, (1). From Lemma 3.1, it follows that if $l - r + 1 < n$ and $d \gg 0$, then the generic degree d form in $n+1$ variables cannot be written in the form (1.1). ☐

Remark 3.3. We can now assume $l - r + 1 = n$. We have already dealt with the case that $n = 1$ or $n = 2$. On the other hand, if $n \geq 3$, we can eliminate tuples (n, l, r, d) with $l \geq n+3$ by Lemma 3.2. So, we are only left with the tuples of the form $(n, n, 1, d)$, $(n, n+1, 2, d)$, and $(n, n+2, 3, d)$ with $d \geq r$. But we have already taken care of the tuples of the form $(n, n, 1, d)$ and $(n, n+1, 2, d)$, so it suffices to determine for which d the Question 1.1 has a positive answer for $(n, n+2, 3, d)$. The remaining sections deal with this case.

4. Interlude: reformulating our question

To complete our proof of Theorem 1.2, it suffices to determine which tuples of the form $(n, n+2, 3, d)$ with $d \geq 3$ satisfy Question 1.1. In the language of star configurations, we wish to know which degree d generic hypersurfaces in $\mathbb{P}^n$ contains a star configuration $\mathbb{X}(n+2) = \mathbb{X}(n+2, 3)$. We make a brief interlude to derive some technical results, moving Question 1.1 back and forth between questions in algebra and questions in geometry.

We first notice the following trivial fact:

Lemma 4.1. *Let $\{F = 0\}$ be an equation of the degree d hypersurface $Y \subset \mathbb{P}^n$. Then Y contains a star configuration $\mathbb{X}(n+2)$ only if there are $L_1, \ldots, L_l$, with $l = n+2$, general linear forms such that*

$$F = \sum_{\sigma = \{i_1, i_2, i_3\} \subseteq [n+2]} L_\sigma M_\sigma$$

where the $\binom{n+2}{3}$ forms M_σ have degree $d-3$.

Hence, it is natural to perform the following geometric construction. We define a map

$$\Phi_{d,l} : \underbrace{S_1 \times \cdots \times S_1}_{l=n+2} \times \underbrace{S_{d-3} \times \cdots \times S_{d-3}}_{\binom{n+2}{3}} \longrightarrow S_d$$

of affine varieties such that

$$\Phi_{d,l}\left(L_1, \ldots, L_l, M_{\{1,2,3\}}, \ldots, M_\sigma, \ldots, \ldots M_{\{n,n+1,n+2\}}\right) = \sum_{\sigma \subseteq [n+2],\ |\sigma|=3} L_\sigma M_\sigma.$$

We then rephrase our question in terms of the map $\Phi_{d,l}$:

Lemma 4.2. *Let $d, l = n+2$ be non-negative integers with $d \geq l-1$. Then the following are equivalent:*

(*i*) *Question 1.1 has an affirmative answer for $(n, l, r, d) = (n, n+2, 3, d)$.*
(*ii*) *the map $\Phi_{d,l}$ is a dominant map.*

Proof. Lemma 4.1 proves that (i) implies (ii). To prove the other direction, it is enough to show that for a generic form F, the fibre $\Phi_{d,l}^{-1}(F)$ contains a set of l linear forms defining a star configuration. More precisely, define $\Delta \subset S_1 \times \cdots \times S_1 \times S_{d-3} \times \cdots \times S_{d-3}$ as follows:

$$\Delta = \left\{ (L_1, \ldots, L_l, \ldots, M_\sigma, \ldots) \;\middle|\; \begin{array}{c} \text{there exists } \sigma = \{a, b, c\} \subseteq [n+2] \text{ such that} \\ L_a, L_b, L_c \text{ are linearly dependent} \end{array} \right\}.$$

Then we want to show that $\Phi_{d,l}^{-1}(F) \not\subset \Delta$.

We proceed by contradiction, assuming that the generic fibre of $\Phi_{d,l}$ is contained in Δ. Then Δ would be a component of the domain of $\Phi_{d,l}$. This is a contradiction as the latter is an irreducible variety being the product of irreducible varieties. □

Using the map $\Phi_{d,l}$ we can now translate Question 1.1 into an ideal theoretic question.

Lemma 4.3. *Let $d, l = n+2$ be non-negative integers with $d \geq l-1$. Consider l generic forms $L_1, \ldots, L_l \in S = \mathbb{C}[x_0, \ldots, x_n]$ and $\binom{n+2}{3}$ forms $\{M_\sigma \in S_{d-3} \mid \sigma \subseteq [n+2] \text{ and } |\sigma| = 3\}$.*

Define the following l forms of degree $d-1$:

$$\begin{aligned}
Q_1 &= \sum_{\sigma \subseteq [n+2],\ 1 \in \sigma} \frac{L_\sigma M_\sigma}{L_1} = \sum_{\{a,b\} \subseteq [n+2] \setminus \{1\}} L_a L_b M_{\{1\} \cup \{a,b\}}, \\
Q_2 &= \sum_{\sigma \subseteq [n+2],\ 2 \in \sigma} \frac{L_\sigma M_\sigma}{L_2}, \\
&\vdots \\
Q_l &= \sum_{\sigma \subseteq [n+2],\ l \in \sigma} \frac{L_\sigma M_\sigma}{L_l}.
\end{aligned}$$

With this notation, form the ideal

$$I = (L_\sigma \mid \sigma \subseteq [l] \text{ and } |\sigma| = 3) + (Q_1, \ldots, Q_l) \subseteq S.$$

Then the following are equivalent:

(i) Question 1.1 has an affirmative answer for $(n, l, r, d) = (n, n+2, 3, d)$;
(ii) $I_d = S_d$.

Proof. Using Lemma 4.2 we just have to show that $\Phi_{d,l}$ is a dominant map if and only if $I_d = S_d$. In order to do this we will determine the tangent space to the image of $\Phi_{d,l}$ at a generic point $q = \Phi_{d,l}(p)$, where $p = (L_1, \ldots, L_l, \ldots, M_\sigma, \ldots)$. We denote with T_q this affine tangent space.

The elements of the tangent space T_q are obtained as

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=0} \Phi_{d,l}\left(L_1 + tL_1', \ldots, L_l + tL_l', M_{\{1,2,3\}} + tM_{\{1,2,3\}}', \ldots, M_\sigma + tM_\sigma', \ldots\right) \\
&\qquad = \left.\frac{d}{dt}\right|_{t=0} \sum_{\sigma=\{i,j,k\} \subseteq [l]} (L_i + tL_i')(L_j + tL_j')(L_k + tL_k')(M_\sigma + tM_\sigma')
\end{aligned}$$

when we vary the forms $L_i' \in S_1$ and $M_\sigma' \in S_{d-3}$. By a direct computation we see that the elements of T_q have the form

$$\sum_{\sigma=\{i,j,k\} \subseteq [l]} [L_i' L_j L_k M_\sigma + L_i L_j' L_k M_\sigma + L_i L_j L_k' M_\sigma + L_i L_j L_k M_\sigma']$$

$$= \left(\sum_{i=1}^{l} L_i' \left(\sum_{i\in\sigma\subseteq[l]} \frac{L_\sigma M_\sigma}{L_i}\right)\right) + \left(\sum_{\sigma\subseteq[l]} L_\sigma M_\sigma'\right).$$

Since the $L_i', L_j', L_k' \in S_1$ and $M_\sigma' \in S_{d-3}$ can be chosen freely, we obtain that $I_d = T_q$. □

Remark 4.4. Lemma 4.3 can be used to computationally provide a positive answer for each tuple of the form $(n, n+2, 3, d)$. To do this, we proceed as follows. Given d and $l = n+2$ we construct the ideal I as described above by choosing forms L_i and M_i. We then compute $\dim_{\mathbb{C}} I_d$ using a computer algebra system. If $\dim_{\mathbb{C}} I_d = \dim_{\mathbb{C}} S_d$, then, by upper semicontinuity, we have proved that Question 1.1 has an affirmative answer for that tuple $(n, n+2, 3, d)$. On the other hand, if we pick L_i's and M_i's such that $\dim_{\mathbb{C}} I_d < \dim_{\mathbb{C}} S_d$ we cannot eliminate $(n, n+2, 3, d)$ since another choice of forms may give equality.

5. Base case: $\mathbb{X}(4)$ in $\mathbb{P}^2$

We now show that the generic degree $d \geq 3$ hypersurface of $\mathbb{P}^2$ contains a star configuration $\mathbb{X}(4)$, i.e., we prove Theorem 1.2 for $(2, 4, 3, d)$ for all $d \geq 3$. Note that this result was already proved in [6], but we give a new proof that better lends itself to our induction argument for proving that the generic degree $d \geq 3$ hypersurface of $\mathbb{P}^n$ for all $n \geq 2$ contains a star configuration $\mathbb{X}(n+2)$.

We first begin with a lemma about matrices that shall prove useful:

Lemma 5.1. *Let $\mathcal{A}_r = (a_{ij})$ be a square $r\times r$ matrix where $r > 1$ and $a_{ij} = \begin{cases} 1 & \text{if } i \neq j; \\ 0 & \text{if } i = j. \end{cases}$, i.e.,*

$$\mathcal{A}_r = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & \ddots & \ddots & 1 \\ \vdots & & \ddots & \ddots & 1 \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix}. \tag{5.1}$$

Then $\mathcal{A}_r$ has maximal rank.

Proof. Consider the square $r \times r$ matrix $U = \mathcal{A}_r + I_r$, i.e. this is a matrix where every element is equal to one. Clearly $\mathrm{rk}(U) = 1$, and the eigenvalues are r, with multiplicity one, and 0, with multiplicity $(r-1)$. Suppose that $\det(\mathcal{A}_r) = 0$. Then there exists an eigenvector $v \neq 0$ such that $\mathcal{A}_r v = 0$, and hence, $(\mathcal{A}_r + I_r)v = \mathcal{A}_r v + I_r v = v$, i.e., U should have 1 as an eigenvalue. From this contradiction, we deduce that $\mathrm{rk}(\mathcal{A}_r) = r$. □

We will use the following notation in the proof given below. Let $d \geq 3$ be a non-negative integer, let $L_1, \dots, L_4 \in S_1$ be four generic linear forms in $S = \mathbb{C}[x_0, x_1, x_2]$ and consider any six forms

$$\{M_\sigma \in S_{d-3} \mid \sigma \subseteq [4] \text{ and } |\sigma| = 3\}.$$

We will abuse notation and write M_{ijk} for $M_{\{i,j,k\}}$. Using these forms, we define the follow four forms of degree $d-1$:

$$\begin{array}{rcl} Q_1 & = & M_{123}L_2L_3 + M_{124}L_2L_4 + M_{134}L_3L_4 \\ Q_2 & = & M_{123}L_1L_3 + M_{124}L_1L_4 + M_{234}L_3L_4 \\ Q_3 & = & M_{123}L_1L_2 + M_{134}L_1L_4 + M_{234}L_2L_4 \\ Q_4 & = & M_{124}L_1L_2 + M_{134}L_1L_3 + M_{234}L_2L_3. \end{array}$$

With this notation, we form the ideal

$$I = (L_1L_2L_3, \ldots, L_2L_3L_4, Q_1, \ldots, Q_4) \subset S. \tag{5.2}$$

Then for $d \geq 3$, we give an affirmative answer to Question 1.1:

Theorem 5.2. *The generic degree $d \geq 3$ curve in $\mathbb{P}^2$ contains a $\mathbb{X}(4)$.*

Proof. Our strategy is to use Lemma 4.3 to show that the ideal I of (5.2) has the property that $I_d = S_d$. In particular, given 4 generic linear forms $L_1, \ldots, L_4$, we need to pick forms M_σ with $\sigma \subseteq [4]$ and $|\sigma| = 3$ so the ideal (5.2) has this desired property.

Because the generators $\{L_\sigma \mid \sigma \subseteq [4] \text{ and } |\sigma| = 3\}$ are the generators of a star configuration $\mathbb{X}(4)$, we know by Theorem 2.5 that for all $d \geq 3$

$$\dim_{\mathbb{C}}(S/(L_{123}, \ldots, L_{234}))_d = 6.$$

If we set $A = S/(L_{123}, \ldots, L_{234})$, it therefore suffices to find 6 linear independent elements in $I/(L_{123}, \ldots, L_{234})$ of degree d. We will prove that the equivalence classes of the following 6 elements in A are linearly independent

$$L_2Q_3, L_1Q_2, L_3Q_1, L_4Q_1, L_4Q_2, L_4Q_2, L_4Q_3 \tag{5.3}$$

for a generic choice of the forms M_σ with $\deg M_\sigma = d-3$. As noted by Remark 4.4, it is enough to show these forms are linearly independent for a special choice of forms for M_σ.

We first construct an evaluation table. For each $\tau = \{i,j\} \subseteq [4]$, let

$$p_{r,s} := V(L_i) \cap V(L_j) \text{ where } \{r,s\} \cup \tau = [4]$$

denote one of the six points of $\mathbb{X}(4)$. We construct the following evaluation table where entry (i,j) is formed by evaluating the polynomial labeling column j at the point labeling row i.

(5.4)

	L_3Q_1	L_1Q_2	L_2Q_3	L_4Q_1	L_4Q_2	L_4Q_3
$p_{1,2}$	0	0	$M_{123}L_1L_2^2$	0	0	0
$p_{1,3}$	0	$M_{123}L_1L_3^2$	0	0	0	0
$p_{2,3}$	$M_{123}L_2L_3^2$	0	0	0	0	0
$p_{1,4}$	0	$M_{124}L_1^2L_4$	0	0	$M_{124}L_1L_4^2$	$M_{134}L_1L_4^2$
$p_{2,4}$	0	0	$M_{234}L_2^2L_4$	$M_{124}L_2L_4^2$	0	$M_{234}L_2L_4^2$
$p_{3,4}$	$M_{134}L_3^2L_4$	0	0	$M_{134}L_3L_4^2$	$M_{234}L_3L_4^2$	0

For example, the entry $(4,2)$ is the polynomial L_1Q_2 evaluated at $p_{1,4}$, that is

$$\begin{array}{rcl} L_1Q_2(p_{1,4}) & = & (M_{123}L_1^2L_3 + M_{124}L_1^2L_4 + M_{234}L_1L_3L_4)(p_{1,4}) \\ & = & (M_{124}L_1^2L_4)(p_{1,4}) \text{ since } L_3(p_{1,4}) = 0. \end{array}$$

With a slight abuse of notation we adopt the following convention that if in the row indexed by $p_{i,j}$ we write $M_{ijk}L_a^cL_b^d$, then this a shorthand form for $(M_{ijk}L_a^cL_b^d)(p_{i,j})$.

Observe that the evaluation matrix (5.4) holds for any choice of $M_\sigma \in S_{d-3}$. For each $d \geq 3$, we want to show one can pick specific M_σ's so that this matrix has rank 6. It would then follow that the forms (5.3) are linearly independent in A, and the conclusion follows.

Note that for any nonzero choice M_{123}, the first three rows of this matrix are linear independent. We will be finished if we can show that the submatrix formed by the last three rows and three columns has maximal rank, i.e, we can find choices for M_{124}, M_{134}, and M_{234} that make the matrix

$$\begin{array}{c|ccc} & L_4Q_1 & L_4Q_2 & L_4Q_3 \\ \hline p_{1,4} & 0 & M_{124}L_1L_4^2 & M_{134}L_1L_4^2 \\ p_{2,4} & M_{124}L_2L_4^2 & 0 & M_{234}L_2L_4^2 \\ p_{3,4} & M_{134}L_3L_4^2 & M_{234}L_3L_4^2 & 0 \end{array} \tag{5.5}$$

have rank three.

When $d = 3$, we set $M_\sigma = 1$ when $4 \in \sigma$. We can therefore factor the above matrix as

$$\begin{bmatrix} L_1L_4^2 & 0 & 0 \\ 0 & L_2L_4^2 & 0 \\ 0 & 0 & L_3L_4^2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The first matrix is clearly invertible, and the second matrix is invertible by Lemma 5.1. Thus the matrix (5.5) has rank three, and thus the entire evaluation matrix (5.4) has maximal rank.

When $d > 3$, we set

$$M_{124} = L_2^{d-3} \qquad M_{234} = L_3^{d-3} \qquad M_{134} = L_4^{d-3}.$$

When we use this choice of M_σ's, the evaluation matrix (5.5) given above becomes

$$\begin{array}{c|ccc} & L_4Q_1 & L_4Q_2 & L_4Q_3 \\ \hline p_{1,4} & 0 & 0 & \star \\ p_{2,4} & \star & 0 & 0 \\ p_{3,4} & \star & \star & 0 \end{array}$$

where $\star$ represents a non-zero value. But then it is immediate that this matrix has rank three, and thus the entire matrix (5.4) has maximal rank. $\square$

6. Induction Step: $\mathbb{X}(n+2)$ in $\mathbb{P}^n$

We now prove the general situation.

Theorem 6.1. *The generic degree $d \geq 3$ hypersurface of $\mathbb{P}^n$ contains a $\mathbb{X}(n+2)$.*

Proof. We work by induction on n. If $n = 2$, then the result is true by Theorem 5.2.

To better understand the induction step, we will show how we pass from the case $n = 2$ to $n = 3$. In $\mathbb{P}^3$, $l = 5$, and thus $\mathbb{X}(5)$ contains 10 points $\{p_{i,j} \mid 1 \leq i < j \leq 5\}$. In

particular for each $\tau = \{i_1, \ldots, i_3\} \subseteq [5]$ with $|\tau| = 3$,

$$p_{r,s} := V(L_{i_1}) \cap \cdots \cap V(L_{i_3}) \text{ where } \{r, s\} \cup \tau = [5] \text{ for general linear forms } L_1, \ldots, L_5.$$

For each $d \geq 3$, we construct an evaluation matrix $\mathcal{M}_3$ of size 10×10 in the following way. Let $Q_1, \ldots, Q_5$ be the forms constructed from $L_1, \ldots, L_5$ as in Lemma 4.3. Our evaluation matrix is then:

$$\mathcal{M}_3 = \begin{array}{c|ccc|cccc} & L_3Q_1 & \cdots & L_4Q_3 & L_5Q_1 & L_5Q_2 & \cdots & L_5Q_4 \\ \hline p_{1,2} & & & & & & & \\ \vdots & & \overline{\mathcal{M}_2} & & & & \mathbf{0} & \\ p_{3,4} & & & & & & & \\ \hline p_{1,5} & & & & & & & \\ \vdots & & \mathcal{F} & & & & \mathcal{G} & \\ p_{4,5} & & & & & & & \end{array}$$

where the matrix $\overline{\mathcal{M}_2}$ is formally the same as the 6×6 matrix constructed in the proof of Theorem 5.2, $\mathbf{0}$ is the 6×4 zero matrix, $\mathcal{F}$ is a 4×6 matrix, and $\mathcal{G}$ is a 4×4 matrix. It should be clear that the top right block is the zero matrix since each point $p_{1,2}, \ldots, p_{3,4}$ vanishes on the line $V(L_5)$.

As in Theorem 5.2, we need to show that we can pick the M_σ's that appear in the construction of $Q_1, \ldots, Q_5$ so that the above evaluation matrix has $\mathrm{rk}(\mathcal{M}_3) = 10$. By induction, we can find M_σ's with $\sigma \subseteq [4]$ and $|\sigma| = 3$ so that the matrix $\overline{\mathcal{M}_2}$ has maximal rank. We will therefore finish the proof for the $n = 3$ case if we can show that the matrix $\mathcal{G}$ has rank 4.

The matrix $\mathcal{G}$ is a 4×4 matrix of type

$$\begin{array}{c|cccc} & L_5Q_1 & L_5Q_2 & L_5Q_3 & L_5Q_4 \\ \hline p_{15} & 0 & M_{125}L_1L_5^2 & M_{135}L_1L_5^2 & M_{145}L_1L_5^2 \\ p_{25} & M_{125}L_2L_5^2 & 0 & M_{235}L_2L_5^2 & M_{245}L_2L_5^2 \\ p_{35} & M_{135}L_3L_5^2 & M_{235}L_3L_5^2 & 0 & M_{345}L_3L_5^2 \\ p_{45} & M_{145}L_4L_5^2 & M_{245}L_4L_5^2 & M_{345}L_4L_5^2 & 0 \end{array}$$

Again, as in the proof of Theorem 5.2, we write $M_{ijk}L_a^bL_c^d$ to mean the value of $M_{ijk}L_a^bL_c^d(p_{r,s})$. Note that no M_σ with $\sigma \subseteq [4]$ appears in the above matrix, so fixing these values in $\overline{\mathcal{M}_2}$ has no impact in $\mathcal{G}$.

When $d = 3$, we set each $M_{ij5} = 1$ when defining $Q_1, \ldots, Q_5$. In this case, we can factor $\mathcal{G}$ as

$$\begin{bmatrix} L_1L_5^2 & 0 & 0 & 0 \\ 0 & L_2L_5^2 & 0 & 0 \\ 0 & 0 & L_3L_5^2 & 0 \\ 0 & 0 & 0 & L_4L_5^2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Both matrices are invertible (we are using Lemma 5.1 for the second matrix), so the matrix $\mathcal{G}$ is invertible, and thus has maximal rank.

When $d > 3$, we set

$$\begin{array}{rclrcl} M_{125} & = & L_2^{d-3} & M_{235} & = & L_3^{d-3} \\ M_{135} & = & L_3^{d-3} & M_{245} & = & L_4^{d-3} \\ M_{145} & = & L_5^{d-3} & M_{345} & = & L_4^{d-3}. \end{array}$$

With these choices, the evaluation matrix $\mathcal{G}$ becomes:

$$\begin{array}{c|cccc} & L_5Q_1 & L_5Q_2 & L_5Q_3 & L_5Q_4 \\ \hline p_{16} & 0 & 0 & 0 & \star \\ p_{26} & \star & 0 & 0 & 0 \\ p_{36} & \star & \star & 0 & 0 \\ p_{46} & \star & \star & \star & 0 \end{array}$$

where $\star$ is a non-zero value. In this form, it is clear that the matrix has maximal rank.

We now describe the general induction step. That is, suppose that $d \geq 3$ and that the theorem is true for $\mathbb{P}^n$. We prove the statement for $\mathbb{P}^{n+1}$.

Let $L_1, \dots, L_{n+3}$ be $n+3$ generic linear forms. For each $\tau = \{i_1, \dots, i_{n+1}\} \subseteq [n+3]$ with $|\tau| = n+1$, we set

$$p_{r,s} := V(L_{i_1}) \cap \cdots \cap V(L_{i_{n+1}}) \text{ where } \{r, s\} \cup \tau = [n+3].$$

To finish the proof, it suffices to show that we can find choices for M_σ as $\sigma \subseteq [n+3]$ with $|\sigma| = 3$ so that the evaluation matrix

$$\mathcal{M}_{n+1} = \begin{array}{c|ccc|cccc} & L_5Q_1 & \cdots & L_{n+2}Q_{n+1} & L_{n+3}Q_1 & L_{n+3}Q_2 & \cdots & L_{n+3}Q_{n+2} \\ \hline p_{1,2} & & & & & & & \\ \vdots & & \overline{\mathcal{M}}_n & & & & \mathbf{0} & \\ p_{n+1,n+2} & & & & & & & \\ \hline p_{1,n+3} & & & & & & & \\ \vdots & & \mathcal{F} & & & & \mathcal{G} & \\ p_{n+2,n+3} & & & & & & & \end{array}$$

has maximal rank. Here, $\overline{\mathcal{M}}_n$ is formally the same matrix as $\mathcal{M}_n$, the matrix $\mathbf{0}$ is an appropriate sized zero matrix, $\mathcal{F}$ is a $(n+2) \times \binom{n+2}{2}$ matrix, and $\mathcal{G}$ is a $(n+2) \times (n+2)$ matrix of the form:

$$\begin{array}{c|ccccc} & L_{n+3}Q_1 & L_{n+3}Q_2 & \cdots & L_{n+3}Q_{n+1} & L_{n+3}Q_{n+2} \\ \hline p_{1,n+3} & 0 & M_{1,2,n+3}L_1L_{n+3}^2 & \cdots & M_{1,n+1,n+3}L_1L_{n+3}^2 & M_{1,n+2,n+3}L_1L_{n+3}^2 \\ p_{2,n+3} & M_{1,2,n+3}L_2L_{n+3}^2 & 0 & \cdots & M_{2,n+1,n+3}L_2L_{n+3}^2 & M_{2,n+2,n+3}L_2L_{n+3}^2 \\ \vdots & \vdots & & \ddots & & \vdots \\ p_{n+1,n+3} & M_{1,n+1,n+3}L_{n+1}L_{n+3}^2 & M_{2,n+1,n+3}L_{n+1}L_{n+3}^2 & \cdots & 0 & M_{n+1,n+2,n+3}L_{n+1}L_{n+3}^2 \\ p_{n+2,n+3} & M_{1,n+2,n+3}L_{n+2}L_{n+3}^2 & M_{2,n+2,n+3}L_{n+2}L_{n+3}^2 & \cdots & M_{n+1,n+2,n+3}L_{n+2}L_{n+3}^2 & 0 \end{array}$$

By induction, we can find M_σ with $\sigma \subseteq [n+3]$ and $|\sigma| = 3$, and $n+3 \notin \sigma$ so that the matrix $\overline{\mathcal{M}}_n$ has maximal rank. It remains to show that $\mathcal{G}$ has maximal rank.

As in the case $n = 3$, when $d = 3$, we set every $M_\sigma = 1$ when $\sigma \subseteq [n+3]$ with $|\sigma| = 3$, and $n+3 \in \sigma$. Using Lemma 5.1, we can show that $\mathcal{G}$ has maximal rank. When $d > 3$,

we set

$$M_{i,j,n+3} = L_j^{d-3} \text{ for all } \sigma \subseteq [n+3] \text{ with } |\sigma| = 3, \text{ and } n+3 \in \sigma$$

except for $M_{1,n+2,n+3}$, which we set to $M_{1,n+2,n+3} = L_{n+3}^{d-3}$. The evaluation matrix $\mathcal{G}$ then becomes

	$L_{n+3}Q_1$	$L_{n+3}Q_2$	$\cdots$	$L_{n+3}Q_{n+1}$	$L_{n+3}Q_{n+2}$
$p_{1,n+3}$	0	0	$\cdots$	0	$\star$
$p_{2,n+3}$	$\star$	0	$\cdots$	0	0
$\vdots$	$\vdots$		$\ddots$		$\vdots$
$p_{n+1,n+3}$	$\star$	$\star$	$\cdots$	0	0
$p_{n+2,n+3}$	$\star$	$\star$	$\cdots$	$\star$	0

where $\star$ represents a non-zero value. Because it is clear that this matrix will have rank $n+2$, this completes the proof. □

We are now able to complete the proof the main theorem:

Proof of Theorem 1.2, (2), case (viii). By Theorem 5.2 and Theorem 6.1, Question 1.1 holds for all tuples of the form $(n, n+2, 3, d)$ with $d \geq 3$. □

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